ABRIDGED — MATHEMATICS FRAMEWORK

for the 2017 NATIONAL ASSESSMENT OF EDUCATIONAL PROGRESS

NATIONAL ASSESSMENT GOVERNING BOARD

INTRODUCTION

Mathematics is an integral part of education, and a firm grasp of its principles and application is essential to thriving in an information-based society. To succeed in school and in life as an adult, children need a strong foundation in mathematics, beginning with traditional skills such as arithmetic and advancing their mathematics literacy to algebra, geometry, calculus, and beyond. U.S. students also need to master the fundamentals of the information age—such as problem solving, communicating mathematical concepts, and applying mathematics in real-world settings—to compete globally.

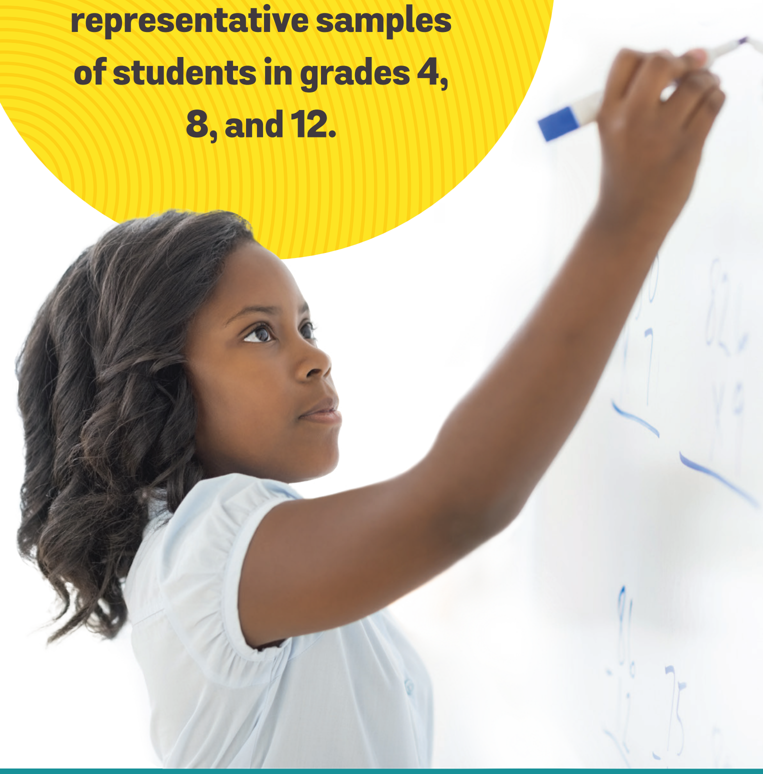
NAEP OVERVIEW

The National Assessment of Educational Progress (NAEP) is the only continuing and nationally representative measure of trends in academic achievement of U.S. elementary and secondary school students in various subjects. For more than four decades, NAEP assessments have been conducted periodically in reading, mathematics, science, writing, U.S. history, civics, geography, and other subjects. By collecting and reporting information on student performance at the national, state, and local levels, NAEP is an integral part of our nation's evaluation of the condition and progress of education. The information NAEP provides about student achievement helps the public, educators, and policymakers understand strengths and weaknesses in student performance and make informed decisions about education.

The National Assessment Governing Board was created by Congress in 1988 to set policy for NAEP. The Board oversees the development and updating of NAEP frameworks that describe the specific knowledge and skills to be assessed in each subject. *The NAEP Mathematics Framework* resulted from the work of many individuals and organizations involved in mathematics, including mathematicians, educators, researchers, policymakers, and members of the public.

The NAEP Mathematics Assessment regularly collects achievement information on representative samples of students in grades 4, 8, and 12. Through The Nation's Report Card, NAEP reports how well students use and apply mathematics knowledge in various situations by answering multiple-choice and constructed-response questions. As required under the No Child Left Behind Act, NAEP assesses mathematics in grades 4 and 8 every two years and in grade 12 every four years. The results provide a rich, broad, and deep picture of student mathematics achievement in the U.S., reported in terms of achievement levels, scale scores, and percentiles. Only academic achievement data and related information are collected about contexts for student learning. The privacy of individual students and their families is protected.

The NAEP Mathematics Assessment regularly collects achievement information on representative samples of students in grades 4, 8, and 12.



HISTORY OF THE NAEP MATHEMATICS FRAMEWORK

The main NAEP assessment is based on a framework, such as this one, that may be updated periodically. In comparison with the framework for the 2005 and earlier assessments, the 2009–2017 NAEP Mathematics Framework introduces a new subtopic of mathematical reasoning at grades 4, 8, and 12, and new clarifications and examples that describe the levels of mathematical complexity.

At grades 4 and 8, the framework and content objectives for measuring mathematics knowledge and skills have been modified somewhat since the early 1990s, allowing trend lines to continue from that time through 2017. New content objectives were introduced at grade 12 in the 2009 Mathematics Framework, however, to identify the essential mathematics knowledge and skills required for college and workplace training.

CONTENT AND DESIGN

To achieve an appropriate balance of content and reflect a variety of ways of understanding and applying mathematics, *The 2017 Mathematics Framework* specifies that assessment questions measure one of five mathematical content areas:

- **Number Properties and Operations**—including computation and understanding of number concepts
- **Measurement**—including use of instruments, application of processes, and concepts of area and volume
- **Geometry**—including spatial reasoning and applying geometric properties
- **Data Analysis, Statistics, and Probability**—including graphical displays and statistics
- **Algebra**—including representations and relationships

The distribution of questions among these five content areas at each grade is a critical feature of the assessment design because it reflects the relative importance and value given to each area on the assessment. See exhibit 2 on framework page 6.

The design of NAEP involves multiple test booklets, with questions distributed across booklets, so that students taking part in the assessment do not all receive the same questions. In NAEP mathematics, students take the assessment for about 50 minutes total, consisting of two 25-minute sets of questions.

The assessment contains sets of questions for which calculators are not allowed, and other blocks that contain some questions that would be difficult to solve without a calculator. At each grade level, about two-thirds of the blocks measure students' mathematical knowledge and skills without access to a calculator; the other one-third allows calculator use. The type of cal-

culator students may use varies by grade level, as follows:

- **Grade 4:** A four-function calculator is supplied to students
- **Grade 8:** A scientific calculator is supplied to students
- **Grade 12:** Students are allowed to bring whatever calculator, graphing or other, they are accustomed to using in the classroom, with some restrictions for test security purposes. For students who do not bring a calculator to use on the assessment, NAEP will provide a scientific calculator.

Mathematical Content Areas

Number Properties and Operations

Numbers let us talk in a precise way about anything that can be counted, measured, or located in space. Number sense, or the comfort in dealing with numbers effectively, is a major expectation of *The NAEP Mathematics Assessment*. Number sense includes firm intuition about what numbers tell us; an understanding of the ways to represent them symbolically (including facility with converting between different representations); the ability to calculate, either exactly or approximately, and by several means (mentally, with paper and pencil, or with calculator); skill in estimation; and the ability to deal with proportion (including percent).

In fourth grade, students are expected to have a solid grasp of whole numbers as represented by the decimal system and to begin to understand fractions. They should be able to perform such tasks as identifying place values and actual value of digits in whole numbers and adding, subtracting, multiplying, and dividing whole numbers.

By eighth grade, they should be comfortable with rational numbers, represented either as decimal fractions (including percents) or as common fractions, and be able to use them to solve problems involving proportionality and rates. At grade 8, students should also have some familiarity with naturally occurring irrational numbers, such as square roots and pi.

By 12th grade, students should be comfortable dealing with all types of real numbers and vari-

ous representations, such as exponents or logarithms. Students at grade 12 should be familiar with complex numbers and should be able to establish the validity of numerical properties using mathematical arguments. See exhibit 3 on framework pages 9-13.

Measurement

Measurement is the process by which numbers are assigned to describe the world quantitatively. This process involves selecting the attribute of the object or event to be measured, comparing this attribute to a unit, and reporting the number of units. Measurement also allows us to model positive and negative numbers as well as irrational numbers. This connection between measuring and number makes measurement a vital part of the school curriculum. Measurement also has a strong connection to other areas of mathematics and other subjects in the school curriculum.

The NAEP Mathematics Framework includes attributes such as capacity, weight/mass, time, and temperature and the geometric attributes of length, area, and volume. At grade 4, the framework emphasizes length, including perimeter, distance, and height. At grade 8, the framework places more emphasis on areas and angles, and by 12th grade, it focuses on volumes, and rates constructed from other attributes, such as speed.

The NAEP assessment includes nonstandard, customary, and metric units. At grade 4, it emphasizes common customary units such as inch, quart, pound, and hour, and common metric units such as centimeter, liter, and gram. The framework for grades 8 and 12 includes the use of square and cubic units for measuring area, surface area, and volume; degrees for measuring angles; and constructed units such as miles per hour. Converting from one unit in a system to another, such as from minutes to hours, is an important aspect of measurement included in problem situations. Understanding and using the many conversions available is an important skill. See exhibit 4 on framework pages 14-17.

Geometry

Geometry began as a practical collection of rules for calculating lengths, areas, and volumes of common shapes, and has developed into the study of the possible structures of space.

By grade 4, students are expected to be familiar with simple figures and their attributes, both in the plane (lines, circles, triangles, rectangles, and squares) and in space (cubes, spheres, and cylinders).

By eighth grade, students gain a deeper understanding of these shapes, with the study of cross-sections of solids and the beginnings of an analytical understanding of properties of plane figures, especially parallelism, perpendicularity, and angle relations in polygons. Schools introduce right angles and the Pythagorean theorem, and geometry becomes more and more mixed with measurement.

By 12th grade, students are expected to make, test, and validate conjectures. Using analytic geometry, the key areas of geometry and algebra merge into a powerful tool that provides a basis for calculus and the applications of math-

ematics that helped create our modern technological world. See exhibit 5 on framework pages 19-23.

Data Analysis, Statistics, and Probability

Data analysis and statistics refer to the process of collecting, organizing, summarizing, and interpreting data. Statistical thinking emphasizes that data analysis should begin not with the data but with a question to be answered. Data should be collected only with a specific question (or questions) in mind and only after a plan (usually called a design) for collecting data relevant to the question. In the context of data analysis or statistics, probability can be thought of as the study of potential patterns in outcomes that have not yet been observed.

By grade 4, students are expected to apply their understanding of number and quantity to pose questions that can be answered by collecting appropriate data. They should be able to organize data in a table or a plot and summarize the essential features of center, spread, and shape both verbally and with simple summary statistics. Simple comparisons can be made between two related data sets. Fourth-grade students should be able to build the basic concept of chance and statistical reasoning into meaningful contexts, such as, "If I draw two names from among the students in the room, am I likely to get two girls?"

At grade 8, students should be able to use a wider variety of data organizing and summarizing techniques. They should also begin to analyze statistical claims through designed surveys and experiments that involve randomization, using simulation as the main tool for making simple statistical inferences. They should begin to use more formal terminology related to probability and data analysis.

Students at grade 12 should be able to use a wide variety of statistical techniques for all phases of data analysis, including a more formal understanding of statistical inference. In addition to comparing data sets, students should recognize and describe possible associations between two variables by looking at two-way tables for categorical variables or scatterplots for measurement variables. They need knowledge of conditional probability to understand the association between variables, which is related to the concepts of independence and dependence. Students at grade 12 also should be able to use statistical models (linear and nonlinear equations) to describe possible associations between measurement variables, and they should be familiar with techniques for fitting models to data.

Algebra

Algebra was pioneered in the Middle Ages to solve equations easily and efficiently by manipulating symbols, rather than by using the earlier geometric methods of the Greeks. The two approaches were eventually united in the analytic geometry of René Descartes.

As algebra became more widely used, its formal structure began to be studied, resulting in the "rules of algebra"—a compact summary of the principles behind algebraic manipulation. Similar thinking produced a simple but flexible concept of function and also led to the development of set theory as a comprehensive background for mathematics.

These two aspects of algebra—as a powerful representational tool and as a vehicle for comprehensive concepts such as function—form the basis for the expectations spanning grades 4, 8, and 12. By grade 4, students should be able to recognize and extend simple numeric patterns as a foundation for a later understanding of function. They can begin to understand the meaning of equality and some of its properties, as well as the idea of an unknown quantity as a precursor to the concept of variables.

By grade 8, representation of functions as patterns, via tables, verbal descriptions, symbolic descriptions, and graphs, can convey the idea of function. Linear functions receive special attention. They connect to the ideas of proportionality and rate, forming a bridge that will eventually link arithmetic to calculus. Other means of finding solutions, including graphing by hand or with a calculator, reinforce symbolic manipulation in the relatively simple context of linear equations.

By grade 12, students should appreciate the rules of algebra as a basis for reasoning. Nonlinear functions, especially quadratic, power, and exponential functions, are introduced to solve real-world problems. Students should become accomplished at translating verbal descriptions of problem situations into symbolic form. Students at grade 12 also should encounter expressions involving several variables, systems of linear equations, and solutions to inequalities.

MATHEMATICAL COMPLEXITY OF TEST QUESTIONS

Mathematical complexity deals with *what* the students are asked to do in a task. It does not take into account how they might undertake it. In the distance formula task, for example, students who had studied the formula might simply reproduce it from memory. Others who could not recall the exact formula might derive it from the Pythagorean theorem, engaging in a different kind of thinking. To read more about the distance formula task, see framework page 37.

Three categories of mathematical complexity—low, medium, and high—form an ordered description of the demands a question may make on a student's thinking. At the low level of complexity, for example, a student may be asked to recall a property. At the moderate level, the student may be asked to make a connection between two properties; at the high level, a student may need to analyze the assumptions made in a mathematical model. Using levels of complexity for the questions allows for a balance of mathematical thinking in the design of the assessment.

The mathematical complexity of a question is not directly related to its format (i.e., multiple choice, short constructed response, or extended constructed response). Questions requiring that the student generate a response tend to make somewhat heavier demands on students than questions requiring a choice among alternatives, but that is not always the case. Any type of question can deal with mathematics of greater or less depth and sophistication.

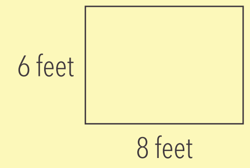
An ideal balance in *The NAEP Mathematics Assessment* for all three grade levels is to devote half of the total testing time to questions of moderate complexity and the remainder divided equally between questions of low and high complexity.

Low-complexity Questions

Low-complexity questions require students to recall or recognize concepts or procedures specified in the framework. These questions typically specify what the student is to do, such as carry out a procedure that can be performed mechanically.

➡ See framework page 38.

SAMPLE QUESTION



1. A teacher drew this rectangle on a playground. Sam walked around the rectangle on the lines shown. How far did Sam walk?

   (A) 14 feet

   (B) 20 feet

   ● 28 feet

   (D) 48 feet

To read the scale score, description of scale score, and performance data of this question, click here and choose score 256.

Moderate-complexity Questions

Questions in the moderate-complexity category involve more flexible thinking and choice among alternatives than do those in the low-complexity category. The student needs to decide what to do and how to do it, bringing together concepts and processes from various domains. For example, the student may be asked to represent a situation in more than one way, to draw a geometric figure that satisfies multiple conditions, or to solve a problem involving multiple unspecified operations. Students might be asked to show or explain their work, but are not expected to justify it mathematically.

➡ See framework page 42.

SAMPLE QUESTION

2. Al, Bev, and Carmen are going on a ride at the park. Only 2 people can go on the ride at a time. They can pair up 3 different ways, as shown below.

   Al and Bev
   Al and Carmen
   Bev and Carmen

   Derek decides to join the group. How many different ways can the 4 students pair up?

To read the scale score, description of scale score, and performance data of this question, click here and choose scores 279.

High-complexity Questions

High-complexity questions make heavy demands on students to use reasoning, planning, analysis, judgment, and creative thought. Students may need to justify mathematical statements, construct a mathematical argument, or generalize from specific examples. Questions at this level take more time than those at other levels due to the demands of the task rather than the number of parts or steps.

➡ See framework page 46.

SAMPLE QUESTION

3. A student was asked to use mathematical induction to prove the following statement.

   $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + (\frac{1}{2})^n = 1 - (\frac{1}{2})^n$

   for all positive integers n.

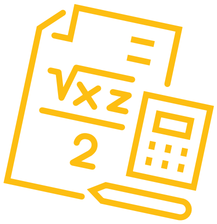
   The beginning of the student's proof is shown below.

   > First, show that the statement is true for $n = 1$:
   >
   > If $n = 1$, $(\frac{1}{2})^1 = 1 - (\frac{1}{2})^1$
   >
   > $\frac{1}{2} = \frac{1}{2}$
   >
   > Next, show that if the statement is true when n is equal to a given positive integer k, then it is also true when n is equal to the next integer, $k + 1$:
   >
   > Assume that the statement is true when $n = k$, so
   >
   > $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + (\frac{1}{2})^k = 1 - (\frac{1}{2})^k$
   >
   > Show that the statement is also true when n is equal to the next integer, $k + 1$.

   Complete the student's proof by showing that if the statement is true when $n = k$, then it is also true when $n = k + 1$ where k is any positive integer.

To read the scale score, description of scale score, and performance data of question 3 on the previous page, click here and choose scores 286.

The careful selection of questions is central to the development of the NAEP Mathematics Assessment.



Question Formats

The careful selection of questions is central to the development of the NAEP Mathematics Assessment. Since 1992, the assessment has used three formats or question types: multiple choice, short constructed response, and extended constructed response. Testing time on NAEP is divided evenly between multiple-choice questions and both types of constructed-response questions, as shown on the next page.

Multiple-Choice Questions

Multiple-choice questions require students to read, reflect, or compute, and then to select the alternative that best expresses the answer. This format is appropriate for quickly determining whether students have achieved certain knowledge and skills. A carefully constructed multiple-choice question can assess any of the levels of mathematical complexity (described earlier), from simple procedures to more sophisticated concepts. Such questions, however, have limited ability to provide evidence of the depth of students' thinking. Multiple-choice questions offer four choices at grade 4 and five choices at grades 8 and 12, and are scored as either correct or incorrect.

SAMPLE QUESTION

4. $\left(16^{1/2}\right)^3 =$

Ⓐ 512

● 64

Ⓒ 48

Ⓓ 24

Ⓔ 12

To read the scale score, description of scale score, and performance data of this question, click here and choose score 193.

Short Constructed-Response Questions

To provide more reliable and valid indications of students' approaches to problems, NAEP assessments include short constructed-response questions. These questions require students to give either a numerical result or the correct name or classification for a group of mathematical objects, draw an example of a given concept, or possibly write a brief explanation for a given result. Short constructed-response questions may be scored correct/incorrect or partially correct, depending on the nature of the problem and the information in students' responses.

SAMPLE QUESTION

5. (a) If c and d are different prime numbers less than 10 and the sum $c + d$ is a <u>composite</u> number greater than 10, what is one possible pair of values for c and d?

   $c =$ ______________

   $d =$ ______________

   (b) If j and k are different prime numbers less than 10 and the sum $j + k$ is a prime number less than 10, what is one possible pair of values for j and k?

   $j =$ ______________

   $k =$ ______________

   (c) If s and t are different prime numbers greater than 10, explain why the sum $s + t$ cannot be a prime number.

To read the scale score, description of scale score, and performance data of this question, click here and choose scores 396, 369, 337, and 311.

Extended Constructed-Response Questions

Extended constructed-response questions require students to consider a situation that demands more than a numerical or short verbal response. These questions have more parts to the response and require more time to complete than other questions. For example, the student may be asked to describe a situation, analyze a graph or table of data, or set up and solve an equation, given a real-world problem. Responses are scored on a five-point scale, ranging from the most complete and mathematically correct response to an incorrect answer. Test yourself: Visit the NAEP Questions Tool.

ACHIEVEMENT LEVELS

Since 1990, the National Assessment Governing Board has used student achievement levels for reporting results of NAEP assessments. The achievement levels represent an informed judgment of "how good is good enough" in the various subjects assessed. Generic policy definitions for achievement at the *Basic*, *Proficient*, and *Advanced* levels describe in very general terms what students at each grade level should know and be able to do on the assessment.

ACHIEVEMENT LEVEL	POLICY DEFINITION
Basic	This level denotes partial mastery of prerequisite knowledge and skills that are fundamental for proficient work at each grade.
Proficient	This level represents solid academic performance for each grade assessed. Students reaching this level have demonstrated competency over challenging subject matter, including subject-matter knowledge, application of such knowledge to real-world situations, and analytical skills appropriate to the subject matter.
Advanced	This level signifies superior performance.

REPORTING NAEP RESULTS

The 2017 NAEP Mathematics Assessment results are based on nationally representative samples of 149,400 fourth graders from 7,840 schools and 144,900 eighth graders from 6,500 schools. Scores are not reported at the individual level. Because the elementary schools participating in NAEP are more likely to exercise the option of including all of their fourth-grade students in the sample, the number of students assessed at grade 4 is larger than grade 8.

Results for the nation reflect the performance of students attending public schools, private schools, Bureau of Indian Education schools, and Department of Defense schools. Results for states and other jurisdictions reflect the performance of students in public schools only and are reported along with the results for public school students in the nation.

NAEP mathematics results for grades 4 and 8 are reported as average scores on a 0–500 scale. Because NAEP scales are developed independently for each subject, scores cannot be compared across subjects. In addition to reporting an overall mathematics score for each grade, NAEP results are reported as percentages of students performing at or above the *Basic* and *Proficient* levels and at the *Advanced* level, as described in the framework.

Scores are also reported at five percentiles to show trends in results for students performing at lower (10th and 25th percentiles), middle (50th percentile), and higher (75th and 90th percentiles) levels.

Item Maps

NAEP item maps are tools that help readers understand student performance. Item maps help to illustrate what students know and can do in NAEP subject areas by positioning descriptions of individual assessment items at different scores along the NAEP scale at each grade level.

For each assessment, example questions are "mapped" onto the NAEP scale for that subject—with more difficult questions at the top of the map and easier questions at the lower part of the map. The item descriptions used in NAEP item maps focus on the knowledge and skills needed to respond successfully to the assessment item. For multiple-choice items, the description indicates the knowledge or skill demonstrated by selection of the correct option. See page 9 for an example of a multiple-choice item. For constructed-response items, the description takes into account the knowledge or skill specified by the different levels of scoring criteria for that item. See page 12 for an example of a constructed-response item.

The location of the questions on the map indicates that students with that score had a high probability of answering the question correctly. Each item map contains the following:

- **A scale specific to the subject.** Mathematics scale ranges from 0-500, depending on the subject.
- **Scale scores from a given assessment.** These represent the scores for students who were likely to answer a question correctly or

to give a complete response. Constructed-response questions for which students could earn partial credit may appear on the map multiple times, once for each level of credit. Constructed-response items are marked with CR on the map.

- **Descriptions** indicating what students need to know or do to answer the question correctly.
- **Content classifications** that refer to the specific skill area of the subject being assessed; for example, in mathematics, the content classification might be algebra or measurement.
- **Achievement level cut scores** that show whether the student is performing at a *Basic*, *Proficient*, or *Advanced* level.

Descriptions for items that have been released to the public are hyperlinked. These items are not used in future assessments. For some subjects and years, no items were released and so no item descriptors are linked.

For more information about how to read the map, score levels, content classification, and student group performance, visit https://www.nationsreportcard.gov/itemmaps and choose *Read more about item mapping.*

For mathematics, there are five content classifications: number properties and operations; measurement; geometry; data analysis, statistics, and probability; and algebra. Number properties and operations is classified with the icon ●. Measurement is classified with the icon ■. Geometry is classified with the icon ▲. Data analysis, statistics, and probability is classified with the icon ▼. Algebra is classified with the icon ◆.

The graphic on the following page shows example scores and the descriptions indicating skills or knowledge students need to have to correctly answer the questions in the eighth-grade 2013 Mathematics Assessment. Scores from the eighth-grade 2013 Mathematics Assessment are categorized into different levels: **0–261:** Below Basic, **262–298:** Basic, **299–332:** Proficient, **333–500:** Advanced.

Eight assessment items in this item map are hyperlinked to further details about the item.

Item Map for NAEP Mathematics Grade 8

ADVANCED	500	436	Interpret a theoretical probability (MC)
		396	Solve problem involving prime numbers (calculator available) – Extended (CR)
		383	Simplify an algebraic expression (MC)
	333	379	Analyze intersection of two shapes (calculator available) (MC)
PROFICIENT	332	329	Given a ratio, solve a problem (calculator available) (MC)
		320	Identify graph of exponential growth (MC)
		316	Determine area of figure, given side length and perimeter (MC)
	299	316	Use algebraic model to estimate height (MC)
BASIC	298	293	Use average (mean) to solve a problem (MC)
		290	Solve problem involving unit conversions (calculator available) – Partial (CR)
		289	Identify place value of specified digit (calculator available) (MC)
	262	286	Use similar triangles to solve problem (MC)
	261	258	Determine population of a city given density and area (calculator available) (MC)
	0	250	Identify result of combining two shapes (MC)

➡ Visit https://www.nationsreportcard.gov/itemmaps to view the full version of item maps

At any given score point, 65 percent of the students (for a constructed-response question), 74 percent of the students (for a four-option multiple-choice question) or 72 percent of the students (for a five-option multiple-choice question) answered that question successfully. For constructed-response questions, responses could be completely or partially correct and therefore a question can map to several points on the scale.

For example, a four-option multiple-choice item in the 2013 NAEP Mathematics Assessment that maps at 307 on the scale would indicate that eighth-grade students with a score of 307 have a 74 percent chance of answering this item correctly. In other words, out of a sample of 100 students who scored 307, 74 would be expected to have answered this question correctly.

CONCLUSION

The Governing Board would like to thank the hundreds of individuals and organizations whose time and talents contributed to this mathematics framework. The Board believes the framework provides a rich and accurate measure of the mathematics comprehension and problem-solving skills that students need for their schooling and for their future adult lives. Development of these mathematics skills is the responsibility of all teachers—not only mathematics teachers but also teachers across the curriculum—and involves the expectations of parents and society.

The Governing Board hopes that this mathematics framework will serve not only as the foundation for the knowledge and skills students need to succeed in mathematics, but also as a catalyst to improve mathematics achievement for the benefit of students themselves and for our nation.

To access the full 2017 NAEP Mathematics Framework, please visit http://nagb.org/publications/frameworks.htm.

The National Assessment Governing Board is an independent, nonpartisan board whose members include governors, state legislators, local and state school officials, educators, business representatives, and members of the general public. Congress created the 26-member Governing Board in 1988 to set policy for the National Assessment of Educational Progress (NAEP).

NATIONAL ASSESSMENT GOVERNING BOARD

For more information on the National Assessment Governing Board, please visit **www.nagb.gov** or call us at **202-357-6938**.